ON MORITA EQUIVALENCE OF GROUP ACTIONS ON LOCALLY C^*-ALGEBRAS

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ABSTRACT. In this paper, we prove that two continuous inverse limit actions α and β of a locally compact group G on the locally C^*-algebras A and B are strongly Morita equivalent if and only if there is a locally C^*-algebra C such that A and B appear as two complementary full corners of C and there is a continuous action γ of G on C which leaves A and B invariant such that $\gamma|_A = \alpha$ and $\gamma|_B = \beta$. This generalizes a result of Combes, *Proc. London Math. Soc.* 49(1984), 289-306.

1. Introduction

Locally C^*-algebras are generalizations of C^*-algebras. Instead of being by a single C^*-norm, the topology on a locally C^*-algebra is defined by a directed family of C^*-seminorms. Such many concepts as group action on a C^*-algebra, crossed product of a C^*-algebra by a group action, Hilbert C^*-module, adjointable module morphism, group action on a Hilbert C^*-module can be defined in a natural way in the context of locally C^*-algebras. The proofs are not always straightforward.

Phillips [11] introduced the notion of action (inverse limit action) of a locally compact group G on a metrizable locally C^*-algebra A and defined the crossed product of A by a continuous inverse limit action of G on A. In [6], we proved a version of the Takai duality theorem for crossed products of locally C^*-algebras by continuous inverse limit actions. The concept of strong Morita equivalence for locally C^*-algebras was introduced in [4]. In [7], we introduce the notion of strong Morita equivalence on the set of group actions on locally C^*-algebras and prove that it is an equivalence relation. Also, we prove that the crossed products of locally C^*-algebras associated with two strongly Morita equivalent continuous inverse limit actions are strongly Morita equivalent.

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This paper is organizes as follows. In Section 2 we recall some facts about Hilbert modules over locally C^*-algebras and (continuous inverse limit) actions of a locally compact group G on a Hilbert module E over a locally C^*-algebra A. In Section 3 we prove that any (continuous inverse limit) action of G on a full Hilbert A-module E induces a (continuous inverse limit) action of G on the linking algebra $\mathcal{L}(E)$ of E, Proposition 3.1. Also we prove that two continuous inverse limit actions α and β of a locally compact group G on the locally C^*-algebras A and B are strongly Morita equivalent if and only if there is a locally C^*-algebra C such that A and B appear as two complementary full corners of C and there is a continuous inverse limit action γ of G on C which leaves A and B invariant such that $\gamma|_A = \alpha$ and $\gamma|_B = \beta$, Theorem 3.5. This generalizes a result of Combes [2] and is a version of Theorem 3.3 in [4] for group actions on locally C^*-algebras.

2. Preliminaries

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_i$ converges to 0 for all continuous C^*-seminorm p on A. The term of "locally C^* -algebra" is due to Inoue [3].

The set $S(A)$ of all continuous C^*-seminorms on A is directed with the order $p \geq q$ if $p\,(a) \geq q\,(a)$ for all $a \in A$. For each $p \in S(A)$, $\ker p = \{a \in A; p(a) = 0\}$ is a two-sided $*$-ideal of A and the quotient algebra $A/\ker p$, denoted by A_p, is a C^*-algebra in the C^*-norm induced by p. The canonical map from A to A_p is denoted by π_p. For $p, q \in S(A)$ with $p \geq q$ there is a canonical surjective morphism of C^*-algebras $\pi_{pq} : A_p \to A_q$ such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$. Then $\{A_p; \pi_{pq}\}_{p,q\in S(A),p\geq q}$ is an inverse system of C^*-algebras and moreover, the locally C^*-algebras A and $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}} A_p$ are isomorphic.

An approximate unit for A is an increasing net of positive elements $\{e_i\}_{i\in I}$ in A such that $p(e_i) \leq 1$ for all $p \in S(A)$ and for all $i \in I$, and $p\,(ae_i - a) + p(e_i a - a) \to 0$ for all $p \in S(A)$ and for all $a \in A$. Any locally C^*-algebra has an approximate unit.

A morphism of locally C^*-algebras is a continuous morphism of $*$ -algebras. Two locally C^*-algebras A and B are isomorphic if there is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalizations of Hilbert C^*-modules by allowing the inner-product to take values in a locally C^*-algebra rather than in a C^*-algebra.

Definition 2.1. *A pre -Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E\times E\to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:*

1. $\langle\xi,\eta\rangle^* = \langle\eta,\xi\rangle$ *for every* $\xi,\eta\in E$;
2. $\langle\xi,\xi\rangle\geq 0$ *for every* $\xi\in E$;
3. $\langle\xi,\xi\rangle\geq 0$ *for every* $\xi\in E; \langle\xi,\xi\rangle = 0$ *if and only if* $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in S(A)}$ where $\overline{p}_E(\xi)=\sqrt{p\left(\langle\xi,\xi\rangle\right)},\xi\in E$.

Any locally C^* -algebra A is a Hilbert A -module in a natural way.

A Hilbert A -module E is full if the linear space $\langle E,E\rangle$ generated by $\{\langle\xi,\eta\rangle\ ,\ \xi,\eta\in E\}$ is dense in A

Let E be a Hilbert A-module. For $p\in S(A)$, $\ker\overline{p}_E=\{\xi\in E;\overline{p}_E(\xi)=0\}$ is a closed submodule of E and $E_p = E/\ker\overline{p}_E$ is a Hilbert A_p-module with $(\xi+\ker\overline{p}_E)\pi_p(a)=\xi a+\ker\overline{p}_E$ and $\langle\xi+\ker\overline{p}_E,\eta+\ker\overline{p}_E\rangle=\pi_p(\langle\xi,\eta\rangle)$. The canonical map from E onto E_p is denoted by σ_p. For $p,q\in S(A)$, $p\geq q$ there is a canonical morphism of vector spaces σ_{pq} from E_p onto E_q such that $\sigma_{pq}(\sigma_p(\xi))=\sigma_q(\xi)$, $\xi\in E$. Then $\{E_p;A_p;\sigma_{pq},\pi_{pq}\}_{p,q\in S(A),p\geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}(\xi_p a_p)=\sigma_{pq}(\xi_p)\pi_{pq}(a_p),\xi_p\in E_p,a_p\in A_p$; $\left\langle\sigma_{pq}(\xi_p),\sigma_{pq}(\eta_p)\right\rangle=\pi_{pq}(\left\langle\xi_p,\eta_p\right\rangle),\xi_p,\eta_p\in E_p$; $\sigma_{pp}(\xi_p)=\xi_p$, $\xi_p\in E_p$ and $\sigma_{qr}\circ\sigma_{pq}=\sigma_{pr}$ if $p\geq q\geq r$, and $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}} E_p$ is a Hilbert A-module which can be identified with E.

The set $L(E)$ of all adjointable A-module morphisms from E into E is a locally C^*-algebra with topology defined by the family of seminorms $\{\widetilde{p}_{L(E)}\}_{p\in S(A)}$, where $\widetilde{p}_{L(E)}(T)=\|(\pi_p)_*(T)\|_{L(E_p,)}$, $T\in L(E)$ and $(\pi_p)_*(T)(\xi+\ker\overline{p}_E)=T(\xi)+\ker\overline{p}_F$, $\xi\in E$. Moreover, $\{L(E_p);\ \ (\pi_{pq})_*\}_{p,q\in S(A),p\geq q}$, where $(\pi_{pq})_*: L(E_p)\to L(E_q)$ is a morphism of C^*-algebras defined by $(\pi_{pq})_*(T_p)(\sigma_q(\xi))=\sigma_{pq}(T_p(\sigma_p(\xi)))$, is an inverse system of C^*-algebras, and $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}} L(E_p)$ can be identified with $L(E)$.

For $\xi,\eta\in E$ we consider the rank one homomorphism $\theta_{\eta,\xi}$ from E into E defined by $\theta_{\eta,\xi}(\zeta)=\eta\left\langle\xi,\zeta\right\rangle$. Clearly, $\theta_{\eta,\xi}\in L(E)$ and $\theta_{\eta,\xi}^*=\theta_{\xi,\eta}$. The linear subspace of $L(E)$ spanned by $\{\theta_{\eta,\xi};\xi,\eta\in E\}$, denoted by $\Theta(E)$, is a two sided $*$-ideal of $L(E)$. The closure of $\Theta(E)$ in $L(E)$ is denoted by $K\left(E\right)$.

Let E be a full Hilbert A -module. Here we recall some facts about the linking algebra of E from [5]

The direct sum $A \oplus E$ of the Hilbert A-modules A and E is a full Hilbert A-module with the action of A on $A \oplus E$ defined by

$$(A \oplus E, A) \ni (a \oplus \xi, b) \mapsto (a \oplus \xi)\, b = ab \oplus \xi b \in A \oplus E$$

and the inner product defined by

$$(A \oplus E, A \oplus E) \ni (a \oplus \xi, b \oplus \eta) \mapsto \langle a \oplus \xi, b \oplus \eta \rangle = a^* b + \langle \xi, \eta \rangle \in A.$$

Moreover, for each $p \in S(A)$, the Hilbert A_p-modules $(A \oplus E)_p$ and $A_p \oplus E_p$ can be identified [8]. Then the locally C^*-algebras $L(A \oplus E)$ and $\lim\limits_{\substack{\leftarrow \\ p \in S(A)}} L(A_p \oplus E_p)$ can be identified [10].

Let $a \in A$, $\xi \in E$, $\eta \in E$ and $T \in K(E)$. The map $L_{a,\xi,\eta,T} : A \oplus E \to A \oplus E$ defined by

$$L_{a,\xi,\eta,T}(b \oplus \zeta) = (ab + \langle \xi, \zeta \rangle) \oplus (\eta b + T(\zeta))$$

is an element in $L(A \oplus E)$. The locally C^*- subalgebra of $L(A \oplus E)$ generated by

$$\{L_{a,\xi,\eta,T}; a \in A, \xi \in E, \eta \in E, T \in K(E)\}$$

is denoted by $\mathcal{L}(E)$ and it is called the linking algebra of E.

By Lemma III 3.2 in [9], we have

$$\mathcal{L}(E) = \lim_{\substack{\leftarrow \\ p \in S(A)}} \overline{(\pi_p)_* (\mathcal{L}(E))},$$

where $\overline{(\pi_p)_* (\mathcal{L}(E))}$ means the closure of the vector space $(\pi_p)_* (\mathcal{L}(E))$ in $L(A_p \oplus E_p)$. Let $p \in S(A)$. From

$$(\pi_p)_* (L_{a,\xi,\eta,T}) = L_{\pi_p(a), \sigma_p(\xi), \sigma_p(\eta), (\pi_p)_*(T)}$$

for all $a, b \in A$, for all $\xi, \eta, \zeta \in E$, and taking into account that $\mathcal{L}(E_p)$, the linking algebra of E_p, is generated by

$$\{L_{\pi_p(a), \sigma_p(\xi), \sigma_p(\eta), (\pi_p)_*(T)}; a \in A, \xi \in E, \eta \in E, T \in K(E)\}$$

since $\pi_p(A) = A_p$, $\sigma_p(E) = E_p$, and $\overline{(\pi_p)_* (K(E))} = K(E_p)$, we conclude that

$$\mathcal{L}(E) = \lim_{\substack{\leftarrow \\ p \in S(A)}} \mathcal{L}(E_p).$$

Moreover, since $\mathcal{L}(E_p) = K(A_p \oplus E_p)$ [12] and the locally C^*-algebras $K(A \oplus E)$ and $\lim\limits_{\substack{\leftarrow \\ p \in S(A)}} K(A_p \oplus E_p)$ can be identified, the linking algebra of E coincides with $K(A \oplus E)$.

Here we recall some facts about actions of a locally compact group G on Hilbert modules from [7].

Let A and B be two locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module.

Definition 2.2. *A morphism of Hilbert modules from E to F is a map $u : E \to F$ with the property that there is a morphism of locally C^*-algebras $\alpha : A \to B$ such that*

$$\langle u(\xi), u(\eta)\rangle = \alpha(\langle \xi, \eta\rangle)$$

for all $\xi, \eta \in E$. An isomorphism of Hilbert modules is a bijective map $u : E \to F$ such that u and u^{-1} are morphisms of Hilbert modules.

If $u : E \to F$ is a morphism of Hilbert modules and $\alpha : A \to B$ is a morphism of locally C^*-algebras such that $\langle u(\xi), u(\eta)\rangle = \alpha(\langle \xi, \eta\rangle)$, then u is a continuous linear map and $u(\xi a) = u(\xi)\alpha(a)$ for all $a \in A$ and for all $\xi \in E$. Moreover, if u is an isomorphism of Hilbert modules and the Hilbert modules E and F are full, then α is an isomorphism of locally C^*-algebras.

For a Hilbert A-module E,

$$\mathrm{Aut}(E) = \{u : E \to E; u \text{ is an isomorphism of Hilbert modules }\}$$

is a group.

Definition 2.3. *Let G be a locally compact group. An action of G on E is a morphism of groups $g \mapsto u_g$ from G to $Aut(E)$.*

The action $g \mapsto u_g$ of G on E is continuous if the map $G \times E \ni (g, \xi) \mapsto u_g(\xi) \in E$ is jointly continuous.

An action $g \mapsto u_g$ of G on E is an inverse limit action if we can write E as an inverse limit of Hilbert C^*-modules $\lim\limits_{\overleftarrow{\lambda\in\Lambda}} E_\lambda$ in such a way that for each $g \in G$, $u_g = \lim\limits_{\overleftarrow{\lambda\in\Lambda}} u_g^\lambda$, where $g \mapsto u_g^\lambda$ is an action of G on E_λ, $\lambda \in \Lambda$.

If $g \mapsto u_g$ is an inverse limit action of G on E, then $E = \lim\limits_{\overleftarrow{\lambda\in\Lambda}} E_\lambda$ and $u_g = \lim\limits_{\overleftarrow{\lambda\in\Lambda}} u_g^\lambda$ for each $g \in G$, where $g \mapsto u_g^\lambda$ is an action of G on E_λ, $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. Since $g \mapsto u_g^\lambda$ is an action of G on E_λ,

$$\left\| u_g^\lambda(\sigma_\lambda(\xi)\right\|_{E_\lambda} = \|\sigma_\lambda(\xi\|_{E_\lambda}$$

for each $\xi \in E$, and for all $g \in G$ [1, 2]. This implies that

$$\overline{p}_\lambda(u_g(\xi)) = \overline{p}_\lambda(\xi)$$

for all $g \in G$ and for all $\xi \in E$.

Let $S(G, A) = \{p \in S(A); \overline{p}_E(u_g(\xi)) = \overline{p}_E(\xi)$ for all $g \in\}$. From these facts, we conclude that $g \mapsto u_g$ is an inverse limit action of G on E, if $S(G, A)$ is a cofinal

subset of $S(A)$. Therefore, if $g \mapsto u_g$ is an inverse limit action of G on E, we can suppose that $u_g = \lim\limits_{\substack{\leftarrow \\ p \in S(A)}} u_g^p$. Moreover, the inverse limit action $g \mapsto u_g$ of G on E is continuous if and only if the actions $g \mapsto u_g^p$ of G on E_p, $p \in S(A)$ are all continuous.

Definition 2.4. *([11]) An action of G on A is a morphism α from G to $Aut(A)$, the set of all isomorphisms of locally C^*-algebras from A to A. The action α is continuous if the function $(g,a) \mapsto \alpha_g(a)$ from $G \times A$ to A is jointly continuous.*

A continuous action α of G on A is an inverse limit action if we can write A as inverse limit $\lim\limits_{\substack{\leftarrow \\ \delta \in \Delta}} A_\delta$ of C^*-algebras in such a way that there are actions α^δ of G on A_δ such that $\alpha_g = \lim\limits_{\substack{\leftarrow \\ \delta \in \Delta}} \alpha_g^\delta$ for all g in G.

Proposition 2.5. *([7]) Let G be a locally compact group and let E be a full Hilbert A -module. Then any action $g \mapsto u_g$ of G on E induces an action $g \mapsto \alpha_g^u$ of G on A such that*

$$\alpha_g^u\left(\langle \xi, \eta \rangle\right) = \langle u_g\left(\xi\right), u_g\left(\eta\right)\rangle$$

for all $g \in G$ and for all $\xi, \eta \in E$ and an action $g \mapsto \beta_g^u$ of G on $K(E)$ such that

$$\beta_g^u\left(\theta_{\xi,\eta}\right) = \theta_{u_g(\xi),u_g(\eta)}$$

for all $g \in G$ and for all $\xi, \eta \in E$. Moreover, if $g \mapsto u_g$ is a continuous inverse limit action of G on E, then the actions of G on A respectively $K(E)$ induced by u are continuous inverse limit actions.

3. Action on the linking locally C^*-algebra of a Hilbert module

Proposition 3.1. *Let G be a locally compact group, let E be a full Hilbert A-module. Any action $g \mapsto u_g$ of G on E induces an action $g \mapsto \gamma_g^u$ of G on the linking algebra $\mathcal{L}(E)$ of E such that*

$$\gamma_g^u\left(L_{a,\xi,\eta,T}\right) = L_{\alpha_g^u(a),u_g(\xi),u_g(\eta),\beta_g^u(T)}$$

for all $a \in A$, $\xi, \eta \in E$ and $T \in K(E)$. Moreover, if $g \mapsto u_g$ is a continuous inverse limit action, then $g \mapsto \gamma_g^u$ is a continuous inverse limit action.

Proof. Let $g \in G$. The map $w_g^u : A \oplus E \to A \oplus E$ defined by

$$w_g^u\left(a \oplus \xi\right) = \alpha_g^u(a) \oplus u_g\left(\xi\right)$$

is a morphism of Hilbert modules, since

$$\begin{aligned} \left\langle w_g^u\left(a \oplus \xi\right), w_g^u\left(b \oplus \eta\right)\right\rangle &= \left\langle \alpha_g^u(a), \alpha_g^u(b)\right\rangle + \langle u_g\left(\xi\right), u_g\left(\eta\right)\rangle \\ &= \alpha_g^u\left(\langle a \oplus \xi, b \oplus \eta\rangle\right) \end{aligned}$$

for all $a, b \in A$ and for all $\xi, \eta \in E$, and α_g^u is an isomorphism of locally C^*-algebras. Moreover, since w_g^u is invertible and $\left(w_g^u\right)^{-1} = w_{g^{-1}}^u$, w_g^u is an isomorphism of Hilbert modules. It is not difficult to check that $g \mapsto w_g^u$ is an action of G on $A \oplus E$.

Let γ^u be the action of G on $K(A \oplus E)$ induced by w^u. Then

$$\gamma_g^u\left(\theta_{a\oplus\xi, b\oplus\eta}\right) = \theta_{w_g^u(a\oplus\xi), w_g^u(b\oplus\eta)} = \theta_{\alpha_g^u(a)\oplus u_g(\xi), \alpha_g^u(b)\oplus u_g(\eta)}$$

for all $a, b \in A$, for all $\xi, \eta \in E$, and for all $g \in G$.

Let $g \in G$, $a \in A$, $\xi, \eta \in E$ and $T \in K(E)$. We will show that

$$\gamma_g^u\left(L_{a,\xi,\eta,T}\right) = L_{\alpha_g^u(a), u_g(\xi), u_g(\eta), \beta_g^u(T)}.$$

For this, let $\{e_i\}_i$ be an approximate unit for A. From

$$\widetilde{p}_{L(A\oplus E)}(L_{a,0,0,0} - \theta_{a\oplus 0, e_i\oplus 0}) \leq p\left(a - ae_i\right)$$

$$\widetilde{p}_{L(A\oplus E)}\left(L_{0,\xi,0,0} - \theta_{e_i\oplus 0, 0\oplus\xi}\right) \leq \overline{p}_E\left(\xi - \xi e_i\right)$$

and

$$\widetilde{p}_{L(A\oplus E)}\left(L_{0,0,\eta,0} - \theta_{0\oplus\eta, e_i\oplus 0}\right) \leq \overline{p}_E\left(\eta - \eta e_i\right)$$

for all $p \in S(A)$ and for all $i \in I$, and taking into account that γ_g^u, α_g^u and u_g are continuous, $ae_i \to a$, $\xi e_i \to \xi$ and $\eta e_i \to \eta$, we conclude that

$$\gamma_g^u\left(L_{a,\xi,\eta,0}\right) = L_{\alpha_g^u(a), u_g(\xi), u_g(\eta), 0}.$$

If $T \in K(E)$, then there is a net $\{\sum\limits_{k\in I_j} \theta_{\xi_k,\eta_k}\}_j$ in $\Theta(E)$ which converges to T. From

$$\widetilde{p}_{L(A\oplus E)}(L_{0,0,0,T} - \sum_{k\in I_j} \theta_{0\oplus\xi_k, 0\oplus\eta_k}) \leq \widetilde{p}_{L(E)}(T - \sum_{k\in I_j}\theta_{\xi_k,\eta_k})$$

for all $p \in S(A)$, and taking into account that γ_g^u and β_g^u are continuous and

$$\sum_{k\in I_j}\gamma_g^u\left(\theta_{0\oplus\xi_k, 0\oplus\eta_k}\right) = \sum_{k\in I_j}\theta_{0\oplus u_g(\xi_k), 0\oplus u_g(\eta_k)} = 0 \oplus \beta_g^u(\sum_{k\in I_j}\theta_{\xi_j,\eta_j})$$

we deduce that

$$\gamma_g^u\left(L_{0,0,0,T}\right) = L_{0,0,0,\beta_g^u(T)}.$$

Thus we have

$$\begin{aligned} \gamma_g^u\left(L_{a,\xi,\eta,T}\right) &= \gamma_g^u\left(L_{a,\xi,\eta,0}\right) + \gamma_g^u\left(L_{0,0,0,T}\right) \\ &= L_{\alpha_g^u(a), u_g(\xi), u_g(\eta), 0} + L_{0,0,0,\beta_g^u(T)} \\ &= L_{\alpha_g^u(a), u_g(\xi), u_g(\eta), \beta_g^u(T)}. \end{aligned}$$

If u is a continuous inverse limit action, then we can suppose that $u_g = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} u_g^p$, where $g \mapsto u_g^p$ is a continuous action of G on E_p for each $p \in S(A)$. Let $p \in S(A)$

and let $g \mapsto w_g^{u^p}$ be the action of G on $A_p \oplus E_p$ induced by u^p. It is not difficult to check that $\left(w_g^{u^p}\right)_p$ is an inverse system of isomorphisms of Hilbert C^*-modules and $g \mapsto \lim\limits_{\leftarrow p} w_g^{u^p}$ is a continuous inverse limit action of G on $A \oplus E$. Moreover, $w_g^u = \lim\limits_{\leftarrow p} w_g^{u^p}$ for each $g \in G$. By Proposition 2.5, the action γ^u of G on $K(A \oplus E)$ induced by w^u is a continuous inverse limit action, and moreover, $\gamma_g^u = \lim\limits_{\leftarrow p} \gamma_g^{u^p}$ for each $g \in G$, where γ^{u^p} is the action of G of $\mathcal{L}(E_p)$ induced by u^p. $\square$

Remark 3.2. *Let G be a locally compact group, let E be a full Hilbert A-module, and let $g \mapsto u_g$ be an action of G on E.*

(1) *Since the map $a \mapsto L_{a,0,0,0}$ from A to $\mathcal{L}(E)$ identifies A with a locally C^*-subalgebra of $\mathcal{L}(E)$ and*

$$\gamma_g^u \left(L_{a,0,0,0}\right) = L_{\alpha_g^u(a),0,0,0}$$

for all $a \in A$ and for all $g \in G$, the restriction of γ^u to A can be identified with the action of G on A induced by u.

(2) *Since the map $T \mapsto L_{0,0,0,T}$ from $K(E)$ to $\mathcal{L}(E)$ identifies $K(E)$ with a locally C^*-subalgebra of $\mathcal{L}(E)$ and*

$$\gamma_g^u \left(L_{0,0,0,T}\right) = L_{0,0,0,\beta_g^u(T)}$$

for all $T \in K(E)$ and for all $g \in G$, the restriction of γ^u to $K(E)$ can be identified with the action of G on $K(E)$ induced by u.

Definition 3.3. *([7]) Let G be a locally compact group and let $g \mapsto \alpha_g$ and $g \mapsto \beta_g$ be two continuous inverse limit actions of G on two locally C^*-algebras A and B. We say that the actions α and β are conjugate if there is a an isomorphism of locally C^*-algebras $\varphi : A \to B$ such that $\alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi$ for each $g \in G$.*

Definition 3.4. *([7]) Let G be a locally compact group and let $g \mapsto \alpha_g$ and $g \mapsto \beta_g$ be two continuous inverse limit actions of G on two locally C^*-algebras A and B. We say that α and β are strongly Morita equivalent if there is a full Hilbert module E over A, and there is a continuous inverse limit action $g \mapsto u_g$ of G on E such that the actions of G on A respectively $K(E)$ induced by u are conjugate with α respectively β.*

Recall that two locally C^*-algebras A and B are two complementary corners in a given locally C^*-algebra C, if there is two projections e and f in the multiplier algebra $M(C)$ of C such that:

(1) $A = eCe$ and $B = fCf$;

(2) $e + f = 1_{M(C)}$;

(3) the locally C^*-subalgebras CeC and CfC of C are dense in C.

The following theorem is a version of Theorem 2.9 [5].

Theorem 3.5. *Let G be a locally compact group and let $g \mapsto \alpha_g$ and $g \mapsto \beta_g$ be two continuous inverse limit actions of G on two locally C^*-algebras A and B. Then the actions α and β are strongly Morita equivalent if and only if there is a locally C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous inverse limit action $g \mapsto \gamma_g$ of G on C such that A and B are invariant to γ and the actions $g \mapsto \gamma_g|_A$ and $g \mapsto \gamma_g|_B$ of G on A respectively B can be identified with α respectively β.*

Proof. First we suppose that α and β are strongly Morita equivalent. Let (E, u) be the pair consisting of a full Hilbert A-module and a continuous inverse limit action of G on E which implements a strong Morita equivalence between α and β. Let $C = \mathcal{L}(E)$ and let γ^u be the action of G on C induced by u. Then A and B are isomorphic with two complementary full corners in C (Theorem 2.9 in [5]) and $g \mapsto \gamma^u_g$ is a continuous inverse limit action of G on C such that identifying A and B with corners in C, $\gamma^u|_A = \alpha$ and $\gamma^u|_B = \beta$ (Remark 3.2).

Conversely, we suppose that there is a locally C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous inverse limit action $g \mapsto \gamma_g$ of G on C such that A and B are invariant to γ and the actions $g \mapsto \gamma_g|_A$ and $g \mapsto \gamma_g|_B$ of G on A respectively B can be identified with α respectively β. By Proposition 2.8 in [5], the locally C^*-algebras A and C are strongly Morita equivalent. Moreover, if e is a full projection in $M(C)$, the multiplier algebra of C, such that $A = eCe$, then the Hilbert A-module Ce implements a strong Morita equivalence between A and C. Let $g \in G$. For each $c \in C$, $\gamma_g(ce) \in Ce$, since $\gamma_g(ce)e = \gamma_g(ce)$. Thus we can consider the linear map $u_g : Ce \to Ce$ defined by $u_g(ce) = \gamma_g(ce)$. Since

$$\begin{aligned} \langle u_g(ce), u_g(de) \rangle &= \left\langle \gamma_g(ce), \gamma_g(de) \right\rangle = \gamma_g\left(ec^*de\right) \\ &= \alpha_g\left(ec^*de\right) = \alpha_g\left(\langle ce, de \rangle\right) \end{aligned}$$

for all $c, d \in C$ and since u_g is invertible and $(u_g)^{-1} = u_{g^{-1}}$, $u_g \in \mathrm{Aut}(Ce)$. It is not difficult to check that $g \mapsto u_g$ is an action of G on Ce. Moreover, since γ is a continuous inverse limit action, u is a continuous inverse limit action.

Since

$$\langle u_g(ce), u_g(de) \rangle = \alpha_g\left(\langle ce, de \rangle\right)$$

for all $g \in G$ and for all $c, d \in C$, $\alpha^u = \alpha$. From

$$\beta_g^u(\theta_{ce,d^*e}) = \theta_{u_g(ce),u_g(d^*e)} = \theta_{\gamma_g(ce),\gamma_g(d^*e)}$$

for all $g \in G$ and for all $c, d \in C$, and taking into account that CeC is dense in C and an element ced in C can be identified with the element θ_{ce,d^*e} in $K(Ce)$, we deduce that the actions β^u and γ are conjugate. Thus we proved that the actions α and γ are strongly Morita equivalent. In the same way we show that the actions β and γ are equivalent, and since the strong Morita equivalence is an equivalence relation [7], the actions α and β are strongly Morita equivalent. □

Using Lemma 5.2 in [11] and Theorem 3.5 we obtain the following corollary.

Corollary 3.6. *Let G be a compact group and let $g \mapsto \alpha_g$ and $g \mapsto \beta_g$ be two continuous actions of G on two locally C^*-algebras A and B. Then the actions α and β are strongly Morita equivalent if and only if there is a locally C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous inverse limit action $g \mapsto \gamma_g$ of G on C such that A and B are invariant to γ and the actions $g \mapsto \gamma_g|_A$ and $g \mapsto \gamma_g|_B$ of G on A respectively B can be identified with α respectively β.*

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